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The Projective Hilbert Space as a Classical Phase Space for Nonrelativistic Quantum Dynamics

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The projective Hilbert space carries a natural symplectic structure which enables one to reformulate quantum dynamics as a classical Hamiltonian one.

KEY WORDS: quantum dynamics; projective Hilbert space; symplectic structure; phase space

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1. INTRODUCTION

The modern differential geometric approach to Hamiltonian mechanics is based on a symplectic manifold (M, ω), where ω is a closed (strongly) nondegenerate 2-form on a (possibly infinite-dimensional) differentiable manifold M. Given a Hamiltonian function H on the *phase space* M, i.e., a sufficiently differentiable function on M that characterizes the mechanical system, the Hamiltonian vector field X_H is determined by

$$\omega(X_H, \,.\,) = dH, \tag{1}$$

and the integral curves γ of the differential equation

$$\dot{x} = X_H(x), \tag{2}$$

$x = \gamma(t)$, describe the time development of the classical Hamiltonian system.

As is well known, ordinary quantum dynamics looks completely different. The phase space M is replaced by a (complex separable) Hilbert space $\mathcal{H}$ and the Hamiltonian function by a (generally unbounded) self-adjoint operator H acting in $\mathcal{H}$. The time development of quantum systems is given by the one-parameter group $\{e^{-iHt}\}_{t\in\mathbb{R}}$ of unitary operators; a pure quantum state $\psi \in \mathcal{H}$, $\|\psi\| = 1$,

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develops according to

$$\psi_t = e^{-iHt}\psi \tag{3}$$

which can be reformulated in terms of the Schrödinger equation

$$\dot{\psi}_t = -iH\psi_t. \tag{4}$$

More precisely, pure quantum states are described by equivalence classes $[\psi]$ of unit vectors $\psi \in \mathcal{H}$, where two unit vectors are equivalent if they differ by a phase factor. Correspondingly, the time development $\Phi_t([\psi])$ of the state $[\psi]$ is given by

$$\Phi_t([\psi]) = [e^{-iHt}\psi]. \tag{5}$$

Whereas Eqs. (3) and (4) are essentially equivalent (up to domain questions), (5) and (3), resp., (5) and (4), are no longer equivalent; in fact, ψ_t can contain a time-dependent phase factor.

The set of the equivalence classes $[\psi]$ is called the *projective Hilbert space* $\mathcal{P}(\mathcal{H})$; $\mathcal{P}(\mathcal{H})$ can be equipped with a symplectic structure and can be considered as a (generally infinite-dimensional) classical phase space. The pure quantum states $[\psi]$ are the points of the phase space $\mathcal{P}(\mathcal{H})$, and their developments in time are curves γ in $\mathcal{P}(\mathcal{H})$:

$$\gamma(t) := \Phi_t([\psi]) = [e^{-iHt}\psi].$$

We are going to show that these curves γ are the solutions of a differential equation of the form (2) where X_H is again a Hamiltonian vector field; X_H is determined by (1) where ω is the symplectic form of $\mathcal{P}(\mathcal{H})$ and the Hamiltonian function is closely related to the Hamiltonian operator. Furthermore, the map $\Phi\colon \mathcal{P}(\mathcal{H}) \times \mathbb{R} \to \mathcal{P}(\mathcal{H})$, $\Phi([\psi], t) := \Phi_t([\psi]) = [e^{-iHt}\psi]$, is a smooth Hamiltonian flow on $\mathcal{P}(\mathcal{H})$. Note that this way we obtain a nonlinear reformulation of quantum dynamics which is usually formulated in linear terms closely related to the Schrödinger equation.

2. THE SYMPLECTIC STRUCTURE OF $\mathcal{P}(\mathcal{H})$

The elements of the projective Hilbert space $\mathcal{P}(\mathcal{H})$ are equivalence classes $[\psi]$ of vectors of $\mathcal{H}$ where it is not necessary to consider ψ as a unit vector. More precisely, call two vectors of $\mathcal{H}^* := \mathcal{H} \setminus \{0\}$ equivalent if they differ by any complex factor, and define $\mathcal{P}(\mathcal{H})$ to be the set of the corresponding equivalence classes. Using the canonical projection $\pi : \mathcal{H}^* \to \mathcal{P}(\mathcal{H})$, $\pi(\psi) := [\psi]$, we can equip $\mathcal{P}(\mathcal{H})$ with a topology by calling a set $O \in \mathcal{P}(\mathcal{H})$ open if $\pi^{-1}(O) \subseteq \mathcal{H}^*$ is open. This way $\mathcal{P}(\mathcal{H})$ becomes a second countable Hausdorff space and π an open continuous mapping (cf., e.g., Bjelaković, 2001). There are other possibilities for the introduction of the topology of $\mathcal{P}(\mathcal{H})$ (Bjelaković, 2001; Bugajski, 1994; Stulpe and Swat, 2001); in this context it turns out that the topology is metrizable and $\mathcal{P}(\mathcal{H})$ can be considered a separable complete metric space.

To make $\mathcal{P}(\mathcal{H})$ a differentiable manifold, we now introduce an atlas of local charts (U_ϕ, f_ϕ) on $\mathcal{P}(\mathcal{H})$ (cf. Klingenberg, 1982). Let $S := \{\phi \in \mathcal{H} \mid \|\phi\| = 1\}$ be the unit sphere of $\mathcal{H}$ and $\{\phi\}^\perp$ the orthocomplement of a vector $\phi \in S$. For every unit vector $\phi \in S$, consider the open set $U_\phi := \{[\psi] \in \mathcal{P}(\mathcal{H}) \mid \langle\psi|\phi\rangle \neq 0\}$ and define a bijective map $f_\phi : U_\phi \to \{\phi\}^\perp$ according to

$$\xi = f_\phi([\psi]) := \frac{1}{\langle\phi|\psi\rangle}\psi - \phi. \tag{6}$$

The vectors $\xi \in \{\phi\}^\perp$ characterize the *rays* $[\psi]$ nonorthogonal to ϕ; geometrically, they correspond to the intersection points of the rays determined by ψ with the hyperplane $\phi + \{\phi\}^\perp = \{\chi \in \mathcal{H} \mid \langle\phi|\chi\rangle = 1\}$. The inverse maps f_ϕ^{-1} are given by

$$[\psi] = f_\phi^{-1}(\xi) = [\phi + \xi] = \pi(\phi + \xi);$$

since the projection π is open and continuous, the maps f_ϕ are homeomorphisms. The open sets U_ϕ cover $\mathcal{P}(\mathcal{H})$. For any $\phi_1, \phi_2 \in S$, the transition maps $f_{\phi_2} \circ f_{\phi_1}^{-1}$ read

$$\left(f_{\phi_2} \circ f_{\phi_1}^{-1}\right)(\xi) = \frac{\phi_1 + \xi}{\langle\phi_2|\phi_1 + \xi\rangle} - \phi_2 \tag{7}$$

where $\xi \in f_{\phi_1}(U_{\phi_1} \cap U_{\phi_2})$.

Considering the complex Hilbert space $\mathcal{H}$ as a real vector space and decomposing the scalar product into its real and imaginary part,

$$\langle\phi \mid \psi\rangle = h(\phi, \psi) + is(\phi, \psi), \tag{8}$$

we obtain a real scalar product h and a strong symplectic form s on $(\mathcal{H}, \mathbb{R})$. Note that $\|\psi\|^2 = \langle\psi|\psi\rangle = h(\psi, \psi)$ and that $s(\phi, \psi) = h(i\phi, \psi)$ ($\langle\,.\,|\,.\,\rangle$ being linear in the second argument). The form s is skew-symmetric and strongly nondegenerate, the latter meaning that the bounded linear map $I : \mathcal{H} \to \mathcal{H}'$, $(I(\phi))(\psi) := s(\phi, \psi)$ is an isomorphism between $(\mathcal{H}, \mathbb{R})$ and its real dual $\mathcal{H}'$.

Considering the subspaces $\{\phi\}^\perp$ also as real Hilbert spaces and observing that, according to (7), the transition maps are obviously infinitely differentiable, we obtain the following result: *By means of the atlas* $\{(U_\phi, f_\phi)\}_{\phi\in S}$, *the projective Hilbert space* $\mathcal{P}(\mathcal{H})$ *becomes a real* C^∞ *manifold modeled over the Hilbert spaces* $(\{\phi\}^\perp, \mathbb{R})$ *which is in general infinite-dimensional.*

The canonical projection $\pi : \mathcal{H}^* \to \mathcal{P}(\mathcal{H})$ becomes a C^∞ map, and the tangent map $T_\psi\pi$ of π at ψ is defined, $T_\psi\pi : T_\psi\mathcal{H}^* \to T_{[\psi]}\mathcal{P}(\mathcal{H})$. Identifying $T_\psi\mathcal{H}^*$ with $(\mathcal{H}, \mathbb{R})$, and using a local chart (U_ϕ, f_ϕ), $\langle\phi|\psi\rangle \neq 0$, we represent π by $f_\phi \circ \pi : \pi^{-1}(U_\phi) \to \{\phi\}^\perp$ and $T_\psi\pi$ by the differential $D_\psi(f_\phi \circ \pi) : \mathcal{H} \to \{\phi\}^\perp$.

From $(f_\phi \circ \pi)(\psi) = f_\phi([\psi])$ and Eq. (6) it follows that

$$D_\psi(f_\phi \circ \pi)v = -\frac{\langle\phi|v\rangle}{\langle\phi|\psi\rangle^2}\psi + \frac{1}{\langle\phi|\psi\rangle}v \tag{9}$$

where $v \in \mathcal{H}$. Moreover, we have the following lemma.

Lemma 2.1.

(a) For each $\psi \in \mathcal{H}^*$, the tangent map $T_\psi\pi : \mathcal{H} \to T_{[\psi]}\mathcal{P}(\mathcal{H})$ is surjective and $\operatorname{Ker} T_\psi\pi = \mathbb{C}\psi$.

(b) The restriction $T_\psi\pi|_{\{\psi\}^\perp} : \{\psi\}^\perp \to T_{[\psi]}\mathcal{P}(\mathcal{H})$ is a continuous isomorphism ($\{\psi\}^\perp$ being the orthocomplement of ψ w.r.t. the complex scalar product). In particular, $T_\psi\pi\big|_{\{\psi\}^\perp}$ is bijective.

Proof: The tangent map $T_\psi\pi$ can be characterized by $D_\psi(f_\phi \circ \pi)$ where furthermore we can choose $\phi = \frac{\psi}{\|\psi\|}$. Eq. (9) then implies

$$D_\psi(f_\phi \circ \pi)v = \frac{1}{\|\psi\|}\left(-\left\langle \frac{\psi}{\|\psi\|}\middle| v\right\rangle \frac{\psi}{\|\psi\|} + v\right) = \frac{1}{\|\psi\|}\big(I - P_{[\psi]}\big)v$$

where $v \in \mathcal{H}$, I is the unit operator on $\mathcal{H}$, and $P_{[\psi]}$ the orthogonal projection onto the subspace $\mathbb{C}\psi = [\psi] \cup \{0\}$. Consequently,

$$D_\psi(f_\phi \circ \pi)v = \frac{1}{\|\psi\|}P_{\{\psi\}^\perp}v \tag{10}$$

where $P_{\{\psi\}^\perp}$ is the orthogonal projection onto the orthocomplement $\{\psi\}^\perp$ of ψ. From (10) we conclude that $D_\psi(f_\phi \circ \pi)\colon \mathcal{H} \to \{\psi\}^\perp$ is surjective and that $\operatorname{Ker} D_\psi(f_\phi \circ \pi) = \mathbb{C}\psi$. Hence, $T_\psi\pi$ is also surjective and $\operatorname{Ker} T_\psi\pi = \mathbb{C}\psi$.

Because $\mathcal{H} = \mathbb{C}\psi \oplus \{\psi\}^\perp$ it follows that the restriction $T_\psi\pi|_{\{\psi\}^\perp}$ is a linear isomorphism. Moreover, since the tangent space $T_{[\psi]}\mathcal{P}(\mathcal{H})$ inherits the topology from every modeling Hilbert space $\{\phi\}^\perp$, $\langle\phi|\psi\rangle \neq 0$, $T_\psi\pi|_{\{\psi\}^\perp}$ is a continuous isomorphism. □

We remark that, according to the lemma, π is a surjective submersion; i.e., π is surjective, for each $\psi \in \mathcal{H}^*$, $T_\psi\pi$ is surjective, and $\operatorname{Ker} T_\psi\pi$ has a closed complement. In the following, we use the abbreviation $\alpha_\psi := T_\psi\pi|_{\{\psi\}^\perp}$.

In terms of a local chart (U_ϕ, f_ϕ), $\langle\phi|\psi\rangle \neq 0$, α_ψ is represented by $D_\psi(f_\phi \circ \pi)\big|_{\{\psi\}^\perp}$; the latter is given by (9) where $v \in \{\psi\}^\perp$ and $D_\psi(f_\phi \circ \pi)v \in \{\phi\}^\perp$. Let $\xi \in \{\phi\}^\perp$ and

$$v := \langle\phi|\psi\rangle P_{\{\psi\}^\perp}\xi.$$

We have $v \in \{\psi\}^\perp$ and, because $\xi \in \{\phi\}^\perp$,

$$\begin{aligned} D_\psi(f_\phi \circ \pi)v &= -\frac{\langle \phi|(I - P_{[\psi]})\xi\rangle}{\langle \phi|\psi\rangle}\psi + \big(I - P_{[\psi]}\big)\xi \\ &= \frac{\big\langle \frac{\psi}{\|\psi\|}\big|\xi\big\rangle\big\langle\phi\big|\frac{\psi}{\|\psi\|}\big\rangle}{\langle\phi|\psi\rangle}\psi + \big(I - P_{[\psi]}\big)\xi \\ &= P_{[\psi]}\xi + \big(I - P_{[\psi]}\big)\xi \\ &= \xi. \end{aligned}$$

Since $D_\psi(f_\phi \circ \pi)|_{\{\psi\}^\perp}$ represents α_ψ and is consequently bijective, we obtain

$$\big(D_\psi(f_\phi \circ \pi)|_{\{\psi\}^\perp}\big)^{-1}\xi = v = \langle\phi|\psi\rangle P_{\{\psi\}^\perp}\xi \tag{11}$$

where $\xi \in \{\phi\}^\perp$. Equation (11) gives a representation of α_ψ^{-1}.

For each $[\psi] \in \mathcal{P}(\mathcal{H})$, we define a skew-symmetric bilinear form on $T_{[\psi]}\mathcal{P}(\mathcal{H}) \times T_{[\psi]}\mathcal{P}(\mathcal{H})$ according to

$$\omega_{[\psi]}(V, W) := \frac{2}{\|\psi\|^2} s\big(\alpha_\psi^{-1}(V), \alpha_\psi^{-1}(W)\big) \tag{12}$$

where s is given by (8) (for ideas related to the introduction of $\omega_{[\psi]}$, cf. Berndt, 1998; Klingenberg, 1982). We have to show that $\omega_{[\psi]}$ is well-defined, i.e., does not depend on the representative $\psi \in \mathcal{H}^*$. Let $V = \alpha_\psi(v)$, $v \in \{\psi\}^\perp$, and let γ be a smooth curve in $\mathcal{H}^*$ satisfying $\gamma(0) = \psi$ and $\dot\gamma(0) = v$. Since $\pi \circ \lambda\gamma = \pi \circ \gamma$ for every $\lambda \in \mathbb{C}$, it follows that

$$\begin{aligned} V = \alpha_\psi(v) = (T_\psi\pi)v = \frac{d}{dt}\pi(\gamma(t))\bigg|_{t=0} &= \frac{d}{dt}\pi(\lambda\gamma(t))\bigg|_{t=0} \\ = (T_{\lambda\psi}\pi)(\lambda v) = \alpha_{\lambda\psi}(\lambda v); & \end{aligned}$$

i.e., $V = \alpha_\psi(v) = \alpha_{\lambda\psi}(\lambda v)$. In consequence, $\alpha_\psi^{-1}(V) = v$ and $\alpha_{\lambda\psi}^{-1}(V) = \lambda v$, and we finally obtain

$$\begin{aligned} \frac{2}{\|\lambda\psi\|^2} s\big(\alpha_{\lambda\psi}^{-1}(V), \alpha_{\lambda\psi}^{-1}(W)\big) &= \frac{2}{|\lambda|^2\|\psi\|^2} s\big(\lambda\alpha_\psi^{-1}(V), \lambda\alpha_\psi^{-1}(W)\big) \\ &= \frac{2}{|\lambda|^2\|\psi\|^2} \operatorname{Im}\big\langle\lambda\alpha_\psi^{-1}(V), \lambda\alpha_\psi^{-1}(W)\big\rangle \\ &= \frac{2}{\|\psi\|^2} s\big(\alpha_\psi^{-1}(V), \alpha_\psi^{-1}(W)\big), \end{aligned}$$

showing that $\omega_{[\psi]}$ is well-defined. By the properties of s (restricted to $\{\psi\}^\perp \times \{\psi\}^\perp$) it is clear that $\omega_{[\psi]}$ is a strongly nondegenerate 2-form on $T_{[\psi]}\mathcal{P}(\mathcal{H})$.

Using a local chart (U_ϕ, f_ϕ) and taking account of (11), $\omega_{[\psi]}$ is represented according to

$$\omega_{[\psi]}(V, W) = \frac{2|\langle\phi|\psi\rangle|^2}{\|\psi\|^2} s\big(\xi, P_{\{\psi\}^\perp}\eta\big) \tag{13}$$

where $\xi, \eta \in \{\phi\}^\perp$ are the representatives of $V, W \in T_{[\psi]}\mathcal{P}(\mathcal{H})$. Equation (13) shows that the 2-form ω on $\mathcal{P}(\mathcal{H})$ is smooth. Since ω is also closed ($d\omega = 0$), ω is a strong symplectic form on $\mathcal{P}(\mathcal{H})$.

3. QUANTUM DYNAMICS AS A CLASSICAL HAMILTONIAN ONE

Let H be the Hamiltonian operator of a quantum system. For simplicity, we assume that the self-adjoint operator H is bounded. The dynamics of pure quantum states $[\psi] \in \mathcal{P}(\mathcal{H})$ is given by

$$\Phi([\psi], t) = [e^{-iHt}\psi] \tag{14}$$

where $\Phi : \mathcal{P}(\mathcal{H}) \times \mathbb{R} \to \mathcal{P}(\mathcal{H})$ is a smooth flow on $\mathcal{P}(\mathcal{H})$. If Φ satisfies a differential equation $\dot{x} = X_H(x)$ where X_H is a vector field on $\mathcal{P}(\mathcal{H})$ and $x = [\psi]$, resp., $x = \gamma(t) = \Phi([\psi], t)$, it follows from

$$\frac{\partial\Phi}{\partial t}(x, t) = X_H(\Phi(x, t))$$

by setting $t = 0$ that

$$X_H(x) = \frac{\partial\Phi}{\partial t}(x, 0). \tag{15}$$

Conversely, if a vector field X_H is defined by a flow according to (15), it follows from

$$\Phi(x, s + t) = \Phi(\Phi(x, t), s)$$

by differentiation w.r.t. s and setting $s = 0$ that

$$\frac{\partial\Phi}{\partial t}(x, t) = \frac{\partial\Phi}{\partial s}(\Phi(x, t), 0),$$

resp.,

$$\frac{\partial\Phi}{\partial t}(x, t) = X_H(\Phi(x, t)).$$

That is, a smooth flow Φ satisfies $\dot{x} = X_H(x)$ with X_H defined by (15).

Hence, we introduce a vector field $X_H : \mathcal{P}(\mathcal{H}) \to T\mathcal{P}(\mathcal{H})$ according to Eqs. (14) and (15). We obtain

$$\begin{aligned} X_H([\psi]) :&= \frac{\partial\Phi}{\partial t}([\psi], 0) = \frac{\partial}{\partial t}[e^{-iHt}\psi]\bigg|_{t=0} = \frac{\partial}{\partial t}\pi(e^{-iHt}\psi)\bigg|_{t=0} \\ &= T_\psi\pi(-iH\psi). \end{aligned}$$

By the lemma of the preceding section, we have $\operatorname{Ker} T_\psi \pi = \mathbb{C}\psi$ and therefore

$$X_H([\psi]) = T_\psi \pi(-iH\psi) = T_\psi \pi \left(-iH\psi + \left\langle \frac{\psi}{\|\psi\|} \middle| iH\psi \right\rangle \frac{\psi}{\|\psi\|} \right)$$

where $-iH\psi + \langle \frac{\psi}{\|\psi\|} | iH\psi \rangle \frac{\psi}{\|\psi\|} \in \{\psi\}^\perp$. In consequence,

$$X_H([\psi]) = \alpha_\psi \left(-iH\psi + \frac{i\langle\psi|H\psi\rangle}{\|\psi\|^2}\psi \right)$$

and

$$\alpha_\psi^{-1}(X_H([\psi])) = -iH\psi + \frac{i\langle\psi|H\psi\rangle}{\|\psi\|^2}\psi.$$

Using the definition (12) of the symplectic form ω, taking any tangent vector $W \in T_{[\psi]}\mathcal{P}(\mathcal{H})$, and writing $W = \alpha_\psi(w)$, $w \in \{\psi\}^\perp$, we obtain

$$\begin{aligned} \omega_{[\psi]}(X_H([\psi]), W) &= \frac{2}{\|\psi\|^2} s\big(\alpha_\psi^{-1}(X_H([\psi])), \alpha_\psi^{-1}(W)\big) \\ &= \frac{2}{\|\psi\|^2} s\left(-iH\psi + \frac{i\langle\psi|H\psi\rangle}{\|\psi\|^2}\psi, w \right). \end{aligned}$$

From $s(i\psi, w) = \operatorname{Im}\langle i\psi|w\rangle = 0$ it follows that

$$\omega_{[\psi]}(X_H([\psi]), W) = \frac{2}{\|\psi\|^2} s(-iH\psi, w);$$

i.e.,

$$\omega_{[\psi]}(X_H([\psi]), W) = \frac{2}{\|\psi\|^2} h(H\psi, w). \tag{16}$$

Now we consider the expectation valued function $\langle H\rangle$,

$$\langle H\rangle([\psi]) := \frac{\langle\psi|H\psi\rangle}{\|\psi\|^2} = (\langle H\rangle \circ \pi)(\psi)$$

where $[\psi] \in \mathcal{P}(\mathcal{H})$ and $\psi \in \mathcal{H}^*$. Taking again any tangent vector $W = \alpha_\psi(w) \in T_{[\psi]}\mathcal{P}(\mathcal{H})$, $w \in \{\psi\}^\perp$, the differential of $\langle H\rangle$ is given by

$$\begin{aligned} d\langle H\rangle([\psi])W &= d\langle H\rangle(\pi(\psi))(\alpha_\psi(w)) = d\langle H\rangle(\pi(\psi))(T_\psi\pi)w \\ &= d(\langle H\rangle \circ \pi)(\psi)w \\ &= \frac{\langle w|H\psi\rangle + \langle\psi|Hw\rangle}{\|\psi\|^2} - \frac{\langle\psi|H\psi\rangle(\langle w|\psi\rangle + \langle\psi|w\rangle)}{\|\psi\|^4}; \end{aligned}$$

i.e., because $H = H^*$ and $\langle w|\psi\rangle = 0$,

$$d\langle H\rangle([\psi])W = \frac{2}{\|\psi\|^2} h(H\psi, w). \tag{17}$$

Comparing Eqs. (16) and (17), we conclude that

$$\omega_{[\psi]}(X_H([\psi]),\ .\ ) = d\langle H\rangle([\psi]) \tag{18}$$

for all $[\psi] \in \mathcal{P}(\mathcal{H})$ (cf. Eq. (1)); that is, X_H is the Hamiltonian vector field corresponding to the Hamiltonian function $\langle H\rangle : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$. By construction of X_H, the time development γ of a quantum state $[\psi]$,

$$\gamma(t) = \Phi([\psi], t) = [e^{-iHt}\psi], \tag{19}$$

satisfies the differential equation

$$\dot{\gamma}(t) = X_H(\gamma(t)) \tag{20}$$

(cf. Eq. (2)) and is uniquely determined by the initial condition $\gamma(0) = [\psi]$. Thus, for bounded Hamiltonian operators, we have achieved a classical Hamiltonian reformulation of quantum dynamics.

Clearly, the Hamiltonian H is in general an unbounded self-adjoint operator. We briefly indicate how our result, summarized by Eqs. (18)–(20), can be transferred to the unbounded case. If H is an unbounded self-adjoint operator, then its domain D_H can be made a Hilbert space by means of the scalar product

$$\langle \phi \mid \psi\rangle_H := \langle \phi \mid \psi\rangle + \langle H\phi \mid H\psi\rangle,$$

$\phi, \psi \in D_H$. The induced norm $\|\,.\,\|_H$ is equivalent to the graph norm and $H : (D_H, \|\,.\,\|_H) \to (\mathcal{H}, \|\,.\,\|)$ becomes a bounded operator. The projective Hilbert space $\mathcal{P}(D_H)$ is, on the one hand, dense in $\mathcal{P}(\mathcal{H})$ and, on the other hand, by means of the new topology induced by $(D_H, \|\,.\,\|_H)$ itself a differentiable manifold; in fact, $\mathcal{P}(D_H)$ is a so-called *manifold domain* (for this concept, cf. Chernoff and Marsden, 1974). Restricting the symplectic form ω to $\mathcal{P}(D_H)$, $\mathcal{P}(D_H)$ is equipped with a symplectic structure such that the flow of the Hamiltonian vector field determined by the Hamiltonian function $[\psi] \mapsto \langle H\rangle([\psi]) := \frac{\langle\psi|H\psi\rangle}{\|\psi\|^2}$, $[\psi] \in \mathcal{P}(D_H)$, coincides with the time development of the states $[\psi]$ according to the Schrödinger equation.

We further remark that, for a unitary operator U, the mapping $F_U : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$, $F_U([\psi]) := [U\psi]$, preserves the symplectic form ω; i.e., F_U is a canonical transformation. This link between unitary operators and canonical transformations enables one to discuss the action of symmetry groups on quantum states in terms of the symplectic manifold $\mathcal{P}(\mathcal{H})$ and thus confirms its interpretation as a phase space.

Our result (18)–(20) was already derived by Cirelli and Lanzavecchia (1984) (cf. also Cirelli et al., 1990). They used essentially the geometry of the complex Kähler manifolds, whereas our derivation is based on the conception of $\mathcal{P}(\mathcal{H})$ as a real manifold and crucially on the lemma of Section 2. For this and some other reasons we think that our approach is closer to the usual conception of a classical phase space. Finally, our reformulation of quantum dynamics on the phase space

$\mathcal{P}(\mathcal{H})$ supplements the results on the reformulation of quantum probability as a reduced classical fuzzy probability theory on $\mathcal{P}(\mathcal{H})$ as given in Beltrametti and Bugajski (1995) or in Stulpe and Swat (2001).

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